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HYPERBOLIC POSETS AND HOMOLOGY STABILITY FOR $O_{n,n}$

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Introduction

Let F be any field, V^{2n} a $2n$-dimensional vector space over F and q_n the quadratic form

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

on V^{2n} (we will call such forms hyperbolic). Let $O_{n,n}$ be the orthogonal group of q_n. We will prove that the map $i_*: H_k(O_{n,n}) \to H_k(O_{n+1,n+1})$ induced by inclusion is an isomorphism if $n \geq 3k+3$.

This paper grew out of an attempt to understand Vogtmann's paper [3]. The result we obtain is similar to Vogtmann's theorem but the proof is much simpler. The simplification ultimately comes from replacing her poset of isotropic subspaces by a poset of hyperbolic ones.

The paper is organized as follows: Section 1 is devoted to defining the hyperbolic poset of a space with quadratic form and to showing that this poset is highly connected. In Section 2 we prove the stabilization theorem for $O_{n,n}$. Throughout the paper we will use mostly the notation from [3].

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Section 1

Let F be a field and V^n an n-dimensional vector space over F. Let q_n be a symmetric, bilinear form on V^{2n} such that in some basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ q_n has matrix

$$\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

We will call V^{2n} a hyperbolic vector space and q_n the hyperbolic form.

1.1. Definition. (i) A subspace W of a hyperbolic vector space V is *isotropic* if the restriction of the hyperbolic form q to W is zero.

(ii) A *Lagrangian* in a hyperbolic vector space V^{2n} is an n-dimensional isotropic subspace of V^{2n}.

1.2. Lemma. *Let (V^{2n}, q_n) be a hyperbolic vector space over F and $H^{2k} \subset V^{2n}$ a hyperbolic subspace of V^{2n}. Then $V^{2n} = H^{2k} \perp H^{2(n-k)}$ for some $H^{2(n-k)}$ hyperbolic subspace of V^{2n}.*

Proof. Let L^n, L^k be Lagrangians respectively for V^{2n} and H^{2k}. Let now $L^{n'} = L^k + (L^n \cap L^{k\perp})$. Then $L^{n'}$ is a Lagrangian in V^{2n} and $L^k \subset L^{n'}$. We know $V^{2n} = H^{2k} \perp W$ where W has dimension $2(n-k)$ and $q_n|_W$ is non-degenerate. But the projection $\mathrm{pr}: V^{2n} \to W$ carries $L^{n'}$ to a Lagrangian in W, so W is hyperbolic (see for example [2, §1] for more details in a more general situation).

Let $\langle a \rangle$ denote the one-dimensional vector space over F spanned by one vector v with a bilinear form $\langle \cdot, \cdot \rangle$ given by $\langle v, v \rangle = a$. Then we have the following lemma:

1.3. Lemma. *Let $V^{2n+1} = H^{2n} \perp \langle a \rangle$, $2 \mid a$, H^{2n} is hyperbolic. Let H^{2k} be a hyperbolic subspace of V^{2n+1}. Then $V^{2n+1} = H^{2k} \perp H^{2(n-k)} \perp \langle a \rangle$ for some hyperbolic $H^{2(n-k)}$.*

Proof. (i) $\chi(F) \neq 2$ and $a \neq 0$; this follows from Witt's theorem [1, Chapter I, Theorem 4.4] because the form is non-degenerate.

(ii) $a = 0$ (hypothesis $\chi(F) = 2$ implies $a = 0$). Then $\mathrm{pr}: V^{2n+1} \to H^{2n}$ carries H^{2k} isomorphically to some hyperbolic subspace $H^{2k'}$ in H^{2n}. By 1.2 we know $H^{2n} = H^{2k'} \perp H^{2(n-k)}$. So $V^{2n+1} = H^{2k'} \perp H^{2(n-k)} \perp \langle a \rangle$. But obviously $H^{2(n-k)}$ is perpendicular to H^{2k}, so V^{2n+1} has decomposition $H^{2k} \perp H^{2(n-k)} \perp \langle a \rangle$.

1.4. Lemma. *Let W^{2n-1} be a $(2n-1)$-dimensional subspace of a hyperbolic space V^{2n}. Let H^{2k} be a hyperbolic subspace of V^{2n}, $n > k$. Then $H^{2k} \cap W^{2n-1}$ contains a hyperbolic subspace of dimension $2(k-1)$. Moreover, if $H^{2k} \not\subset W^{2n-1}$, then there exists a basis in $H^{2k} \cap W^{2n-1}$ such that in this basis q_n has the matrix*

$$\begin{bmatrix} 0 & I_{k-1} & 0 \\ I_{k-1} & 0 & 0 \\ 0 & 0 & a \end{bmatrix} \quad \textit{where } a \in F \textit{ and } 2 \mid a.$$

Proof. Let $H^{2k} = L^k \oplus \bar{L}^k$ where $L^k, \bar{L}^k$ are Lagrangians in H^{2k} and $\bar{L}^k \simeq (L^k)^*$. Then $\dim(L^k \cap W^{2n-1}) \geq k-1$. Let $M^{k-1} \subset L^k \cap W^{2n-1}$; then $\dim(\bar{M}^{k-1} \cap W^{2n-1}) \geq k-2$ where $\bar{M}^{k-1}$ means any subspace of H^{2k} such that:

(i) $\bar{M}^{k-1}\cap M^{k-1}=0$; (ii) $q|_{\bar{M}^{k-1}\oplus M^{k-1}}$ is hyperbolic.

So we can choose a basis $e_1,\dots,e_k,f_1,\dots,f_k$ in H^{2k} such that $H^{2k}\cap W^{2n-1}$ contains $\mathrm{span}(e_1,\dots,e_{k-1},f_1,\dots,f_{k-2})$. If $H^{2k}\subset W^{2n-1}$, then there is nothing to do. Let $\dim(H^{2k}\cap W^{2n-1})=2k-1$. Let $e_1,\dots,e_{k-1},f_1,\dots,f_{k-2},v_1,v_2$ be a basis for $H^{2k}\cap W^{2n-1}$. Then we can assume that

$$v_1=a_1e_k+b_1f_{k-1}+c_1f_k,$$

$$v_2=a_2e_k+b_2f_{k-1}+c_2f_k.$$

Of course, we can assume $a_2=0$. If $b_1=b_2=0$, then $\mathrm{span}(v_1,v_2)=\mathrm{span}(e_k,f_k)$, so we are done.

If $b_2=0$ and $b_1\neq 0$, then $v_1-(c_1/c_2)v_2=a_1e_k+b_1f_{k-1}$ is isotropic, so

$$\mathrm{span}\left(e_1,\dots,e_{k-1},f_1,\dots,f_{k-2},\frac{1}{b_1}\left(v_1-\frac{c_1}{c_2}v_2\right)\right)$$

is hyperbolic and the desired basis is

$$e_1,\dots,e_{k-1},f_1,\dots,f_{k-2},\frac{1}{b_1}\left(v_1-\frac{c_1}{c_2}v_2\right),f_k-\frac{a_1}{b_1}e_{k-1}.$$

If $b_2\neq 0$, then $\mathrm{span}(e_1,\dots,e_{k-1},f_1,\dots,f_{k-2},(1/b_2)v_2)$ is hyperbolic and we are done as previously. The element a is even because for any $x\in H$, $2\mid q_n(x,x)$.

1.5. Lemma. *Let W^{2n+1} be a vector space over F with a form q which has in some basis the matrix*

$$\begin{bmatrix} 0 & I_n & 0\\ I_n & 0 & 0\\ 0 & 0 & a\end{bmatrix},\quad a\in F \text{ and } 2\mid a.$$

Let V^{2n},H^{2k} be two hyperbolic subspaces of W^{2n+1}. Then $V^{2n}\cap H^{2k}$ contains a $2(k-1)$-dimensional hyperbolic subspace. Moreover, if $H^{2k}\not\subset V^{2n}$, then $q|V^{2n}\cap H^{2k}$ has in a some basis the matrix

$$\begin{bmatrix} 0 & I_{k-1} & 0\\ I_{k-1} & 0 & 0\\ 0 & 0 & b\end{bmatrix},\quad b\in F, 2\mid b.$$

Proof. Do precisely the same as in the proof of Lemma 1.4.

1.6. Definition. Let V^n be a vector space over F and q a symmetric bilinear form on V^n., Then $X(V^n)$ is the poset of proper hyperbolic subspaces of V^n ordered by inclusion. $|X(V^n)|$ means the geometric realization of this poset.

1.7. Theorem. *Let V^n be a vector space over F and q be a symmetric bilinear form on V^n which is hyperbolic if $n=2k$ and which has in some basis the matrix*

$$\begin{bmatrix} 0 & I_k & 0 \\ I_k & 0 & 0 \\ 0 & 0 & a \end{bmatrix}$$

if $n=2k+1$, $a\in F$, $2\,|\,a$. *Then:*

(a) $X(V^{2k})$ *is* $(k-2)$*-spherical,* $k>1$;

(b) $X(V^{2k+1})$ *is* $(k-1)$*-spherical,* $k\geq 1$.

Proof. *Step* I: $k=1$. We have to prove that $X(V^3)$ is 0-spherical, which means that there are at least two hyperbolic subspaces of V^3. Let q have the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{bmatrix}$$

in the basis $e, f, v, 2\,|\,a$. Then:

(i) If $a=0$, then $V_1=\operatorname{span}(e,f)$ and $V_2=\operatorname{span}(e+v, f+v)$ are two distinct hyperbolic subspaces of V^3.

(ii) If $a\neq 0$, then $V_1=\operatorname{span}(e,f)$ and $V_2=\operatorname{span}(-(2/a)e, e-(a/2)f+v)$ are two distinct hyperbolic subspaces.

Step II. $k=2$. Then $X(V^4)$ is obviously 0-spherical.

Step III. Assume that (a) is true for $k\leq n$, and (b) is true for $k<n$.

Claim. $X(V^{2n+1})$ is $(n-1)$-spherical.

Proof of the claim. Let $W^{2n}\subset V^{2n+1}$, W^{2n} is hyperbolic. Define

$$X_0=X(W^{2n})\cup\{W^{2n}\}$$

$|X_0|$ is contractible because $|X_0|$ is homotopic to the cone over $|X(W^{2n})|$. Now for $i\geq 1$ define posets

$$X_i=X_{i-1}\cup\{Z\in X(V^{2n+1});\ Z\not\subset W, \dim(Z)=2n+2-2i\}.$$

Let $Z\in X_i\setminus X_{i-1}$. Then, if $i>1$, we have

$$\begin{aligned} |\operatorname{link}(Z)\cap X_{i-1}| &\simeq |\{U\in X(V^{2n+1});\ U\subset Z\cap W\}| * |\{U\in X(V^{2n+1});\ U\supset Z\}| \\ &\simeq |X(V^{2n+1-2i})| * |X(V^{2i-1})| \qquad \text{(by Lemmas 1.2–1.5)} \\ &\simeq \bigvee S^{n-i-1} * \bigvee S^{i-2} \simeq \bigvee S^{n-2} \qquad \text{(by induction).} \end{aligned}$$

If $i=1$, then

$$|\operatorname{link}(Z)\cap X_0| \simeq |\{U\in X(V^{2n+1});\ U\subset Z\cap W\}| \simeq \bigvee S^{n-2} \quad \text{by Lemma 1.5.}$$

So finally we have $|X_i|\simeq\bigvee S^{n-1}$. But $X_n=X(V^{2n+1})$, so we are done.

Step IV. Assume that (a) is true for $k<n$, and (b) is true for $k<n$.

Claim: (a) is true for $k=n$.

Proof of the claim. Let $e_1,\dots,e_n,f_1,\dots,f_n$ be a hyperbolic basis for V^{2n}. Let $W=\operatorname{span}(e_1,\dots,e_n,f_1,\dots,f_{n-1})$. Define posets:

$$X_0 = X(W),$$
$$X_i = X_{i-1} \cup \{Z \in X(V^{2n});\ Z \not\subset W, \dim(Z) = 2n-2i\}, \quad i>0.$$

X_0 is $(n-2)$-spherical by induction and if $i=1$, $Z \in X_1 \setminus X_0$, we have

$$|\mathrm{link}(Z) \cap X_0| \simeq |\{U \in X(V^{2n});\ U \subset Z \cap W\}| \simeq \bigvee S^{n-3}$$

by Lemma 1.4, so is contractible in $|X_0|$. Generally for $i>1$ and $Z \in X_i \setminus X_{i-1}$, we have

$$\begin{aligned}|\mathrm{link}(Z) \cap X_{i-1}| &\simeq |\{U \in X(V^{2n});\ U \subset Z \cap W\}| * |\{U \in X(V^{2n});\ U \supset Z\}| \\ &\simeq |X(V^{2n-2i-1})| * |X(V^{2i})| \qquad \text{(by Lemmas 1.2–1.5)} \\ &\simeq \bigvee S^{n-i-2} * \bigvee S^{i-2} \simeq \bigvee S^{n-3}, \qquad \text{so is contractible in } |X_{i-1}|.\end{aligned}$$

So $|X_i| \simeq \bigvee S^{n-2}$. But $X_{n-1} = X(V^{2n})$, so the proof is complete.

1.8. Definition. Let $X_k(V^n)$ be a subposet of $X(V^n)$ consisting of elements of dimension $\le 2k$.

1.9. Theorem. *With the same hypothesis as in Theorem* 1.7, *we have*

$$|X_k(V^n)| \simeq \bigvee S^{k-1}, \quad k<n/2.$$

Proof. The proof of this theorem is similar to the proof of Theorem 1.7, so we will skip some details. Generally we will go by induction on k and $n>k$.

Step I. If $k=1$, then the statement of the theorem is obvious for any $n>2$.

Step II. If

$$k = \begin{cases} n/2 - 1 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd,} \end{cases}$$

then we are in the case of Theorem 1.7.

Step III. Assume we know that $|X_i(V^n)| \simeq \bigvee S^{i-1}$ for any $n \ge 2i+1$ and $i<k$. Assume that $V^{n-1} \subset V^n$ and $|X_k(V^{n-1})| \simeq \bigvee S^{k-1}$. We want to show $|X_k(V^n)| \simeq \bigvee S^{k-1}$. Define the following posets:

$$X_{-1} = X_k(V^{n-1}),$$
$$X_i = X_{i-1} \cup \{Z \in X_k(V^n);\ Z \not\subset V^{n-1}, \dim(Z) = 2k-2i\}.$$

We want to show $|X_i| \simeq \bigvee S^{k-1}$ for any $-1 \le i < k$. Let $Z \in X_i \setminus X_{i-1}$. We have:

If $i=0$, then

$$\begin{aligned}|\mathrm{link}(Z) \cap X_{-1}| &\simeq |\{U \in X_k(V^n);\ U \subset Z \cap V^{n-1}\}| \\ &\simeq |X(Z \cap V^{n-1})| \simeq \bigvee S^{k-2}, \qquad \text{so is contractible in } |X_{-1}|.\end{aligned}$$

If $i>0$, then

$$
\begin{aligned}
|\mathrm{link}(Z)\cap X_{i-1}| &\simeq |\{U\in X_k(V^n);\ U\subset Z\cap V^{n-1}\}| * |\{U\in X_k(V^n);\ U\supset Z\}| \\
&\simeq |X(Z\cap V^{n-1})| * |X_i(Z^{\perp})| \qquad \text{(by Lemmas 1.2–1.5 and the induction hypothesis on } k) \\
&\simeq \bigvee S^{k-i-2} * \bigvee S^{i-1} \\
&\simeq \bigvee S^{k-2}, \qquad \text{so is contractible in } |X_{i-1}| \simeq \bigvee S^{k-1}.
\end{aligned}
$$

Hence the proof is complete because $X_{k-1}=X_k(V^n)$.

Section 2

Let V^{2n}, q_n be a hyperbolic vector space over F. From now on we will write V,q instead of V^{2n},q_n if it is clear what is the dimension of the space. If there are no coefficients in homology groups, it means we want them to be integers.

Consider now more closely the poset $X(V)$. We have the filtration in $X(V)$ namely $\emptyset\subset X_1(V)\subset X_2(V)\subset\cdots\subset X_{n-1}(V)=X(V)$ and every $X_i(V)$ is spherical. Therefore the spectral sequence associated to this filtration collapses and we have an exact sequence

$$
(*) \qquad 0\to H_{n-2}(X(V))\to H_{n-2}(X_{n-1}(V),X_{n-2}(V))\to\cdots\to H_1(X_2(V),X_1(V)) \to H_0(X_1(V))\to Z\to 0.
$$

It is easy to see that

$$
\frac{X_{i+1}(V)}{X_i(V)} \simeq \bigvee_{\substack{A \text{ hyperbolic}\\ \dim(A)=(i+1)2}} \mathrm{susp}\,|\mathrm{link}(A)\cap X_i(V)|.
$$

But $|\mathrm{link}(A)\cap X_i(V)|\simeq |X_i(A)| = |X(A)|\simeq \bigvee S^{i-1}$, so we have

$$
H_i(X_{i+1}(V),X_i(V)) = \bigoplus_{\substack{A \text{ hyperbolic}\\ \dim(A)=(i+1)2}} \alpha(A)
$$

where $\alpha(A)=H_{i-1}(X(A))$ if $i>0$ and $\alpha(A)=Z$ if $i=0$.

Obviously the filtration $\emptyset\subset X_1(V)\subset\cdots\subset X_{n-1}(V)$ is invariant under the action of $O_{n,n}$ induced from the action on V.

Let $0\to C_n\to C_{n-1}\to\cdots\to C_0\to 0$ denote the exact sequence $(*)$ and let $E_*O_{n,n}:=E_*$ be a free $Z[O_{n,n}]$-resolution of Z. The double complex $E_*\otimes_{Z[O_{n,n}]} C_*$ gives us a spectral sequence with $E^1_{p,q}=H_q(O_{n,n};C_p)$ and this spectral sequence converges to 0 (since E_* is free and C_* is exact). The action of $O_{n,n}$ on $X_i(V)$ is transitive for every i. Let $e_1,\dots,e_n,f_1,\dots,f_n$ be the standard basis for V^{2n} and let $A_p=\mathrm{span}(e_1,\dots,e_p,f_1,\dots,f_p)$. Then for $1\le p<n$ we have:

$$
E^1_{p,q}=H_q(O_{n,n};C_p)=H_q\Bigg(O_{n,n};\bigoplus_{\substack{A \text{ hyperbolic}\\ \dim(A)=2p}} \alpha(A)\Bigg)
$$

$$=H_q\left(O_{n,n};\, Z[O_{n,n}] \underset{O_{p,p}\times O_{n-p,n-p}}{\otimes} \alpha(A_p)\right)$$

$$=H_q(O_{p,p}\times O_{n-p,n-p};\, \alpha(A_p)).$$

So finally we have proved the following theorem:

2.1. Theorem. *There is a spectral sequence converging to O with* $E^1_{0,q}=H_q(O_{n,n};\, Z)$ *and*

$$E^1_{p,q}=H_q(O_{p,p}\times O_{n-p,n-p};\, \alpha(A_p)) \quad \textit{for } 1\le p<n.$$

The standard inclusion $V^{2(n-1)}\hookrightarrow V^{2n}$ induces the map of spectral sequences for $O_{n-1,n-1}$ and $O_{n,n}$. Taking the mapping cone of this map we get a relative version of 2.1.

2.2. Theorem. *There is a spectral sequence converging to* 0 *with*

$$E^1_{p,q}=H_q(O_{p,p}\times (O_{n-p,n-p}, O_{n-p-1,n-p-1});\, \alpha(A_p)) \quad \textit{for } 1\le p<n-1,$$

$$E^1_{0,q}=H_q(O_{n,n}, O_{n-1,n-1};\, Z).$$

Now we are able to prove the main theorem.

2.3. Theorem. $H_k(O_{n,n}, O_{n-1,n-1})=0$ *for* $n\ge 3k+1$.

Proof. We will go by induction with respect to k. For $k=0$ the theorem is obvious. Assume that

$$H_i(O_{n,n}, O_{n-1,n-1})=0 \quad \text{for } i<k, n\ge 3i+1.$$

Want to show $H_k(O_{n,n}, O_{n-1,n-1})=0$ if $n\ge 3k+1$.

Claim. Assume $n\ge 3k$. Then $E^1_{p,q}=0$ for $p+q\le k+1$, $p\ge 2$ (so $q<k$) where $E^1_{p,q}$ is the spectral sequence from 2.2.

Proof of the claim.

$$E^1_{p,q}=H_q(O_{p,p}\times (O_{n-p,n-p}, O_{n-p-1,n-p-1});\, \alpha(A_p)).$$

By Künneth formula it is enought to show that

$$H_j(O_{n-p,n-p}, O_{n-p-1,n-p-1})=0 \quad \text{for } p+q\le k+1,\ p\ge 2,\ j\le q.$$

But $3j+p+1\le 3q+p+1\le 2q+k+2\le 3k\le n$ so $3j+1\le n-p$ and by induction we are done.

By the Claim we obtain that $d_1: E^1_{1,k}\to E^1_{0,k}$ is an epimorphism if $n\ge 3k$. But by Künneth formula and induction we know

$$E^1_{1,k}=H_k(O_{1,1}\times (O_{n-1,n-1}, O_{n-2,n-2});\, \alpha(A_1))\simeq H_k(O_{n-1,n-1}, O_{n-2,n-2}).$$

Now we need some more notation. Let $e_1,\dots,e_n,f_1,\dots,f_n$ be the basis for V^{2n}

and $e_1', \ldots, e_{n-1}', f_1', \ldots, f_{n-1}'$ the basis for V^{2n-2}. Then let $l: V^{2n-2} \to V^{2n}$ denote the 'lower' inclusion: $l(e_i') = e_{i+1}$, $l(f_i') = f_{i+1}$ for $i = 1, \ldots, n-1$ and $u: V^{2n-2} \to V^{2n}$ denote the 'upper' inclusion: $u(e_i') = e_i$, $u(f_i') = f_i$ for $i = 1, \ldots, n-1$. Notice that $l_* = u_*$ on $H_s(O_{n-1,n-1})$ for any s. By the claim we know that the map $j_n: H_k(O_{n-1,n-1}, O_{n-2,n-2}) \to H_k(O_{n,n}, O_{n-1,n-1})$ induced by the 'lower' inclusions is an epimorphism. Consider now the following commutative diagram (we use the fact that $u_* = 1_*$ on nonrelative homology groups):

$$\begin{tikzcd}
 & H_k(O_{n-2,n-2}, O_{n-3,n-3}) \arrow[r, "\partial"] \arrow[d, "j_{n-1}"] & H_{k-1}(O_{n-3,n-3}) \arrow[r, "u_*"] \arrow[d, "1_*"] & H_{k-1}(O_{n-2,n-2}) \arrow[d, "1_*"] \arrow[dl, equal] \\
H_k(O_{n-1,n-1}) \arrow[r, "f"] \arrow[d, "1_*"] & H_k(O_{n-1,n-1}, O_{n-2,n-2}) \arrow[r, "\partial"] \arrow[d, "j_n"] & H_{k-1}(O_{n-2,n-2}) \arrow[r, "u_*"] \arrow[d, "1_*"] & H_{k-1}(O_{n-1,n-1}) \arrow[dl, equal] \\
H_k(O_{n,n}) \arrow[r] & H_k(O_{n,n}, O_{n-1,n-1}) \arrow[r, "\partial"] & H_{k-1}(O_{n-1,n-1}) &
\end{tikzcd}$$

If $n \geq 3k+1$, then $n-1 \geq 3k$, so by the claim j_{n-1} and j_n are epimorphisms. But if j_{n-1} is an epimorphism, then f is an epimorphism, so $H_k(O_{n,n}, O_{n-1,n-1})$ must be 0. This finishes the proof of the theorem.

2.4. Remark. Let V_{2n} be a $2n$-dimensional vector space over F, q^n a skew-symmetric form on V_{2n} which has the matrix

$$\begin{vmatrix} 0 & I \\ -I & 0 \end{vmatrix}$$

in some basis $e_1, \ldots, e_n, f_1, \ldots, f_n$. Let Sp_n be the orthogonal group of q^n. We will call V_{2n} a symplectic vector space, q^n a symplectic form and Sp_n a symplectic group. Then precisely the same proof works for the standard inclusion $\mathrm{Sp}_{n-1} \hookrightarrow \mathrm{Sp}_n$ (I am very grateful to the referee for pointing out this possibility to me). You have only to replace the words:

symmetric by skew-symmetric, hyperoblic by symplectic

q_n by q^n, V^{2n} by V_{2n}, $O_{n,n}$ by Sp_n.

The symplectic case is even easier because you don't have to worry about the case $a \neq 0$ (see Lemmas 1.3, 1.4, 1.5) – for any $v \in V_{2n}$, $q^n(v, v) = 0$. By the same reason, (i) from the step I of the proof of Theorem 1.7 is obvious without justification. So finally we can say: the standard inclusion $\mathrm{Sp}_{n-1} \hookrightarrow \mathrm{Sp}_n$ induces an isomorphism on k-th homology groups provided $n \leq 3k+3$.

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